# An Application of Group Theory to Music

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## Abstract

In this paper, some concepts of modular arithmetic and group theory are firstly introduced. Then, some musical terms which can be understood without musical background are firstly introduced. Moreover, it is described how to create a twelve-tone row chart using modular arithmetic. Finally, mathematical group theory is applied in some music theory.

**Keywords:** Group, Notes, Chords

## Introduction

Virtually all scientific advances rely on some form of mathematics which is trying to understand conceptual and logical truth, and appreciating the intrinsic beauty. We have known that mathematics is the most abstract of the sciences and music is the most abstract of the arts. The arts largely refer to the human experience. But both mathematics and music are built around abstract patterns. Additional research has shown that human brains are marvelous at pattern matching and pattern predicting, and these abilities are at the core of both mathematics and music.

## Some Concepts on Modular Arithmetic

**Definition**

A relation $\sim$ on a nonempty set $A$ is an **equivalence relation** if the following conditions are satisfied for arbitrary $x, y, z$ in $A$:

(i) $x \sim x$ (reflexive property);

(ii) if $x \sim y$, then $y \sim x$ (symmetric property);

(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$ (transitive property).

**Definition**

Let $n$ be a positive integer greater than 1. For integers $x$ and $y$, $x$ is **congruent to** $y$ **modulo** $n$ if and only if $x - y$ is a multiple of $n$. We write

$$x \equiv y \pmod{n}.$$

**Theorem**

The relation of congruence modulo $n$ is an equivalence relation on $\mathbb{Z}$.

**Proof:**

Let $n > 1$, and $x, y$ and $z$ be arbitrary in $\mathbb{Z}$.

(i) $x \equiv x \pmod{n}$ since $x - x = (n)(0)$.

(ii) $x \equiv y \pmod{n}$ implies $x - y = nq$ for some $q \in \mathbb{Z}$.

Thus $y - x = n(-q)$ and $-q \in \mathbb{Z}$,

$$y \equiv x \pmod{n}.$$

(iii) $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$ imply

$$x - y = nq \text{ and } y - z = nk \text{ for some } q, k \in \mathbb{Z}.$$

Thus $x - z = x - y + y - z = nq + nk = n(q + k)$ and $q + k \in \mathbb{Z}$,

$$x \equiv z \pmod{n}.$$ □

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**Definition**

The equivalence classes for congruence modulo $n$ form a partition of $\mathrm{Z}$, that is, they separate $\mathrm{Z}$ into mutually disjoint subsets. These subsets are called **congruence classes** or **residue classes**. There are $n$ distinct congruence classes modulo $n$ such that

$$
\begin{aligned}
[0] &= \{\ldots,-2n,-n,0,n,2n,\ldots\}\\
[1] &= \{\ldots,-2n+1,-n+1,1,n+1,2n+1,\ldots\}\\
[2] &= \{\ldots,-2n+2,-n+2,2,n+2,2n+2,\ldots\}\\
&\vdots\\
[n-1] &= \{\ldots,-n-1,-1,n-1,2n-1,3n-1,\ldots\}.
\end{aligned}
$$

Let $\mathrm{Z}_n$ denote this set of classes:

$$\mathrm{Z}_n = \{[0],[1],[2],\ldots,[n-1]\}.$$

**Example**

When $n=12$, these classes appear as

$$
\begin{aligned}
[0] &= \{\ldots,-24,-12,0,12,24,\ldots\}\\
[1] &= \{\ldots,-23,-11,1,13,25,\ldots\}\\
[2] &= \{\ldots,-22,-10,2,14,26,\ldots\}\\
&\vdots\\
[11] &= \{\ldots,-13,-1,11,23,35,\ldots\}.
\end{aligned}
$$

**Theorem**

(i) The rule

$$[x]+[y]=[x+y]$$

defines an addition that is a binary operation on $\mathrm{Z}_n$.

(ii) Addition is associative in $\mathrm{Z}_n$:

$$([x]+[y])+[z]=[x]+([y]+[z]).$$

(iii) Addition is commutative in $\mathrm{Z}_n$:

$$[x]+[y]=[y]+[x].$$

(iv) $\mathrm{Z}_n$ has the additive identity $[0]$.

(v) Each $[x]$ in $\mathrm{Z}_n$ has $[-x]$ as its additive inverse in $\mathrm{Z}_n$.

**Proof:**

See [3]. □

**Theorem**

(i) The rule

$$[x][y]=[xy]$$

defines a multiplication that is a binary operation on $\mathrm{Z}_n$.

(ii) Multiplication is associative in $\mathrm{Z}_n$:

$$([x][y])[z]=[x]([y][z]).$$

(iii) Multiplication is commutative in $\mathrm{Z}_n$:

$$[x][y]=[y][x].$$

(iv) $\mathrm{Z}_n$ has the multiplicative identity $[1]$.

**Proof:**

See [3]. $\square$

**Definition**

An integer $d$ is the **greatest common divisor** of $x$ and $y$ if

(i) $d$ is a positive integer,

(ii) $d \mid x$ and $d \mid y$,

(iii) $c \mid x$ and $c \mid y$ imply $c \mid d$.

**Definition**

The two integers $x$ and $y$ are **relatively prime** if their greatest common divisor is 1.

**Theorem**

An element $[x]$ of $\mathbb{Z}_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $x$ and $n$ are relatively prime.

**Proof:**

See [3]. $\square$

**Corollary**

Every nonzero element of $\mathbb{Z}_n$ has a multiplicative inverse if and only if $n$ is a prime.

**Proof:**

See [3]. $\square$

## Some Concepts and Definitions on Music

**Definitions**

A **musical tone** is the quality of sound which is the result of a regular vibration transmitted through the air as a sound wave. The **pitch** of a tone is the frequency of the vibration.

Note that the frequency is usually measured in cycles per second, or **hertz** (Hz). The range of audibility for the human ear is about 20Hz to 20,000Hz. However, we will associate a positive real number $x$ with the frequency $x$ Hz, in order that the set of pitches is in one to one correspondence with the set $\mathbb{R}^+$.

**Definitions**

The specific pitches are called **notes**. The **interval** between two notes is the distance between their two associated pitches.

Table 1. Expression of notes and their frequencies

| Note | Frequency (Hz) | Note | Frequency (Hz) |
|---|---|---|---|
| $C_4$ | 261.63 | $F\sharp_4, G\flat_4$ | 370.00 |
| $C\sharp_4, D\flat_4$ | 277.19 | $G_4$ | 392.00 |
| $D_4$ | 293.67 | $A\flat_4, G\sharp_4$ | 415.31 |
| $E\flat_4, D\sharp_4$ | 311.13 | $A_4$ | 440.00 |
| $E_4$ | 329.63 | $B\flat_4, A\sharp_4$ | 466.17 |
| $F_4$ | 349.23 | $B_4$ | 493.89 |

The twelve pitches of modern system are named by the first 7 letters of the alphabet. Although each letter represents a different frequency, the letters are repeated when the frequency of a pitch is doubled. The range of these pitches is known as an **octave**.

The octave can be divided into 12 equal intervals in order that the frequency of each pitch results from multiplying the previous one by $\sqrt[12]{2}$. This is known as **equal tempered tuning**.

**Definitions**

The difference in frequency between each note is called a **semitone**. The symbol ♯ which is called sharp, is used to denote a pitch that is a semitone above the original and the symbol ♭ which is called flat, to denote a pitch that is a semitone below the original, and a natural ♮ cancels the effect of a sharp or a flat.

**Definitions**

The set of twelve notes is said to be **chromatic scale** and it is musically denoted by

$$E, F, F\sharp, G, G\sharp, A, A\sharp, B, C, C\sharp, D, D\sharp.$$

Since two successive notes differ by a semitone, the note which is a semitone above $G$ is denoted by $G\sharp$. But this note is also a semitone below $A$, it can also be denoted as $A\flat$. This property of the notes which have multiple names in equal tempered tuning is known as **harmonic equivalence**.

As all multiples of a certain frequency are represented by the same letter, it is mathematically convenient to represent the set of twelve notes by the integer modulo 12, where each element is a class and represents an infinite set of numbers.

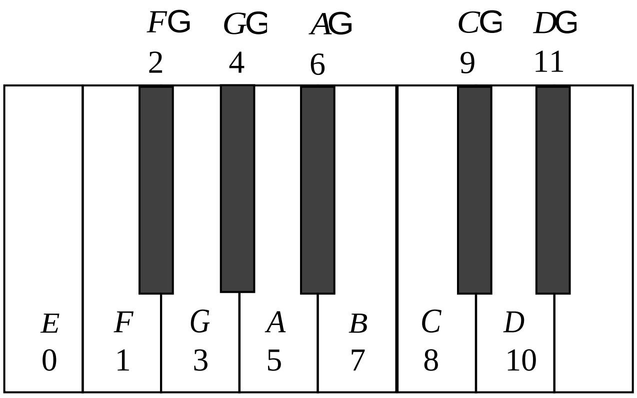


Figure 1. Notes assigned to the elements of $Z_{12}$

## Creating an *n*-Tone Row Chart by Using Modular Arithmetic

We are able to create *n*-tone row chart by using modular arithmetic. Let $n \in \mathbb{Z}^+$ and $a_1, a_2, \ldots, a_n$ be an original row such that

$$a_1 = [0], a_2, \ldots, a_n$$

from $\mathbb{Z}_n$. Then we will make the $n \times n$ row chart by taking

$$\text{entry } (i, j) = a_j - a_i.$$

**Twelve-Tone Row Chart**

A twelve-tone composition is based on a row chart, which is a 12 by 12 array having the following properties:

(i) Each entry is one of 12 note-classes modulo octave.

(ii) Each row and each column contains each note class precisely once.

(iii) All entries can be obtained from the top row as follows.

The leftmost column is the **inversion** of the top row. That is, the interval from the top left note class to the $n^{\text{th}}$ entry in the left column is the opposite of the interval from the top left note class to the $n^{\text{th}}$ entry in the top row.

The subsequent rows are **transpositions** of the top row. That is, they are obtained by starting with the left entry which has been provided above and transposing the first row. Thus, the intervals from entry 1 to entry $m$ in the $n^{\text{th}}$ row is the same as the interval from entry 1 to entry $m$ in the first row.

Then, the columns will be transpositions of the inversion of the original row, or, equivalently, inversions of the various transpositions of the original row. The number of possible original rows is

$$12! = 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1.$$

**Example**

Consider the original row

$$E \quad C \quad F \quad D \quad D\text{G} \quad C\text{G} \quad B \quad A\text{G} \quad G \quad F\text{G} \quad A \quad G\text{G}.$$

Let $E$ be our designated note class. According to modular interval from $E$, the sequence is

$$[0] \quad [8] \quad [1] \quad [10] \quad [11] \quad [9] \quad [7] \quad [6] \quad [3] \quad [2] \quad [5] \quad [4].$$

Let the sequence be started with $[0]$. Then, by replacing each entry in the sequence by its additive inverse or negative, the inversion of the given row is obtained as follows.

$$[0] \quad [4] \quad [11] \quad [2] \quad [1] \quad [3] \quad [5] \quad [6] \quad [9] \quad [10] \quad [7] \quad [8].$$

Next, we label the entries of the original row as:

$$a_1 = [0] \quad a_2 = [4] \quad a_3 = [11] \quad a_4 = [2] \quad a_5 = [1] \quad a_6 = [3]$$
$$a_7 = [5] \quad a_8 = [6] \quad a_9 = [9] \quad a_{10} = [10] \quad a_{11} = [7] \quad a_{12} = [8].$$

Giving by the negatives in $Z_{12}$, the first column will be the inversion such that

$$-a_1 = [0] \quad -a_2 = [8] \quad -a_3 = [1] \quad -a_4 = [10] \quad -a_5 = [11] \quad -a_6 = [9]$$
$$-a_7 = [7] \quad -a_8 = [6] \quad -a_9 = [3] \quad -a_{10} = [2] \quad -a_{11} = [5] \quad -a_{12} = [4].$$

The $(i, j)^{\text{th}}$ entry should make the interval $a_j$ with leftmost entry in the $i^{\text{th}}$ row, which is $-a_i$. Thus, by filling $a_j - a_i$ in the position of the $(i, j)^{\text{th}}$ entry, we obtain the following row chart.

Table 2. Twelve-tone row chart with modular integers

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| [0] | [8] | [1] | [10] | [11] | [9] | [7] | [6] | [3] | [2] | [5] | [4] |
| [4] | [0] | [5] | [2] | [3] | [1] | [11] | [10] | [7] | [6] | [9] | [8] |
| [11] | [7] | [0] | [9] | [10] | [8] | [6] | [5] | [2] | [1] | [4] | [3] |
| [2] | [10] | [3] | [0] | [1] | [11] | [9] | [8] | [5] | [4] | [7] | [6] |
| [1] | [9] | [2] | [11] | [0] | [10] | [8] | [7] | [4] | [3] | [6] | [5] |
| [3] | [11] | [4] | [1] | [2] | [0] | [10] | [9] | [6] | [5] | [8] | [7] |
| [5] | [1] | [6] | [3] | [4] | [2] | [0] | [11] | [8] | [7] | [10] | [9] |
| [6] | [2] | [7] | [4] | [5] | [3] | [1] | [0] | [9] | [8] | [11] | [10] |
| [9] | [5] | [10] | [7] | [8] | [6] | [4] | [3] | [0] | [11] | [2] | [1] |
| [10] | [6] | [11] | [8] | [9] | [7] | [5] | [4] | [1] | [0] | [3] | [2] |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| [7] | [3] | [8] | [5] | [6] | [4] | [2] | [1] | [10] | [9] | [0] | [11] |
| [8] | [4] | [9] | [6] | [7] | [5] | [3] | [2] | [11] | [10] | [1] | [0] |

We draw the modular clock as follows:

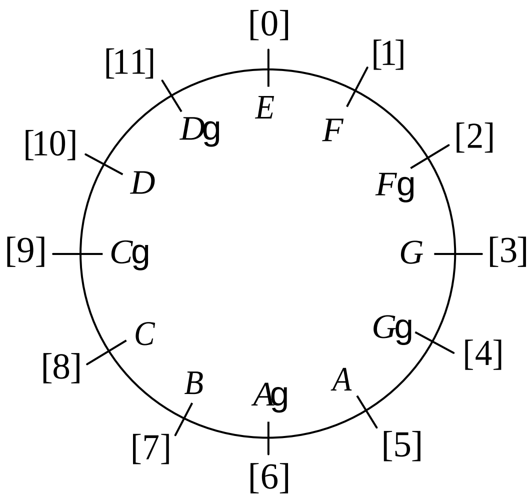


Figure 2. Modular clock

From Table 2 and Figure 2, we obtain Table 3.

Table 3. Twelve-tone row chart with note classes

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| E | C | F | D | DG | CG | B | AG | G | FG | A | GG |
| GG | E | A | FG | G | F | DG | D | B | AG | CG | C |
| DG | B | E | CG | D | C | AG | A | FG | F | GG | G |
| FG | D | G | E | F | DG | CG | C | A | GG | B | AG |
| F | CG | FG | DG | E | D | C | B | GG | G | AG | A |
| G | DG | GG | F | FG | E | D | CG | AG | A | C | B |
| A | F | AG | G | GG | FG | E | DG | C | B | D | CG |
| AG | FG | B | GG | A | G | F | E | CG | C | DG | D |
| CG | A | D | B | C | AG | GG | G | E | DG | FG | F |
| D | AG | DG | C | CG | B | A | GG | F | E | G | FG |
| B | G | C | A | AG | GG | FG | F | D | CG | E | DG |
| C | GG | CG | AG | B | A | G | FG | DG | D | F | E |

**Example**

We will make a seven-tone composition. Let the original row be

$$B \quad G \quad E \quad F \quad C \quad A \quad D.$$

Let $a_1 = [0], a_2 = [5], a_3 = [3], a_4 = [4], a_5 = [1], a_6 = [6], a_7 = [2]$. By using the above method, we have the following row chart:

Table 4. Seven-tone row chart with modular integers

| | | | | | | |
|---|---|---|---|---|---|---|
| [0] | [5] | [3] | [4] | [1] | [6] | [2] |
| [2] | [0] | [5] | [6] | [3] | [1] | [4] |
| [4] | [2] | [0] | [1] | [5] | [3] | [6] |
| [3] | [1] | [6] | [0] | [4] | [2] | [5] |
| [6] | [4] | [2] | [3] | [0] | [5] | [1] |
| [1] | [6] | [4] | [5] | [2] | [0] | [3] |
| [5] | [3] | [1] | [2] | [6] | [4] | [0] |

Table 5. Seven-tone row chart with note classes

| | | | | | | |
|---|---|---|---|---|---|---|
| *B* | *G* | *E* | *F* | *C* | *A* | *D* |
| *D* | *B* | *G* | *A* | *E* | *C* | *F* |
| *F* | *D* | *B* | *C* | *G* | *E* | *A* |
| *E* | *C* | *A* | *B* | *F* | *D* | *G* |
| *A* | *F* | *D* | *E* | *B* | *G* | *C* |
| *C* | *A* | *F* | *G* | *D* | *B* | *E* |
| *G* | *E* | *C* | *D* | *A* | *F* | *B* |

## Some Concepts on Group Theory

**Definition**

Let the binary operation $*$ be defined for elements of a nonempty set $G$. Then $G = (G, *)$ is a **group** with respect to $*$ if the following conditions hold:

(i) $G$ is closed under $*$. That is, $x \in G, y \in G$ imply that $x * y \in G$.

(ii) $*$ is associative. That is, $(x * y) * z = x * (y * z)$ for every $x, y, z \in G$.

(iii) $G$ has an identity element $e$. That is, there is an $e$ in $G$ such that $x * e = e * x = x$ for all $x \in G$.

(iv) $G$ contains inverses. That is, for each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$.

**Definition**

Let $G$ be a group with respect to $*$. Then $G$ is called a **commutative group** or **abelian group** if $*$ is commutative, that is, $x * y = y * x$ for every $x, y \in G$.

**Example**

Let $G = \{e, x, y, z\}$ with multiplication, $\cdot$ , as defined by the following table.

Table 6.

| $\cdot$ | $e$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|
| $e$ | $e$ | $x$ | $y$ | $z$ |
| $x$ | $x$ | $y$ | $z$ | $e$ |
| $y$ | $y$ | $z$ | $e$ | $x$ |
| $z$ | $z$ | $e$ | $x$ | $y$ |

From the table 6, we observe the following:

(i) $G$ is closed under this multiplication.

(ii) $e$ is the identity element.

(iii) Each of $e$ and $y$ its own inverse, and $z$ and $x$ are inverses of each other.

(iv) This multiplication is commutative.

Hence $G$ is an abelian group.

**Example**

The table 7 defines a binary operation $*$ on the set $G = \{a,b,c,d\}$.

Table 7.

| $*$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | $c$ | $a$ | $b$ |
| $b$ | $c$ | $d$ | $b$ | $a$ |
| $c$ | $a$ | $b$ | $c$ | $d$ |
| $d$ | $a$ | $b$ | $d$ | $d$ |

From the table 7, we observe the following:

(i) $G$ is closed under $*$.

(ii) $c$ is the identity element.

(iii) $d$ does not have an inverse since $d * x = c$ has no solution.

Hence $G$ is not a group.

**Example**

The set $Z_n = \{[0],[1],[2],\ldots,[n-1]\}$ of congruence classes modulo $n$ forms an abelian group with respect to addition.

**Theorem**

Every group $G$ satisfies the following properties:

(i) The identity element is unique.

(ii) For every $x \in G$, the inverse $x^{-1}$ is unique.

**Proof:**

See [3]. □

## Some Concepts on Mathematical Music Theory

**Definitions**

The sets of notes are known as **chords**. Chords which contain exactly three notes, modulo octave, are called **triads**. These notes are said to be the root, the third and the fifth, respectively. Since the triads are sets, the order is not important except to identify the root although each triad has the name of its root.

**Definition**

The chord $\{a,b,c\}$ where $a,b,c \in \mathrm{Z}_{12}$ is said to be a **major chord** if $b=a+4$ and $c=a+7$.

**Definition**

The chord $\{a,b,c\}$ where $a,b,c \in \mathrm{Z}_{12}$ is said to be a **minor chord** if $b=a+3$ and $c=a+7$.

**Definition**

Let $\mathrm{M}$ be the set of all chords (major and minor). That is,

$$\mathrm{M} = \{\{a,a+3,a+7\},\{A,A+4,A+7\} \mid a,A \in \mathrm{Z}_{12}\}.$$

An element $x$ in $\mathrm{M}$ is a triad, where $x=\{a,b,c\}$ and $a,b,c \in \mathrm{Z}_{12}$. Since $\mathrm{Z}_{12}=\{[0],\ldots,[11]\}$, $a \in \mathrm{Z}_{12}$ means that $[a] \in \mathrm{Z}_{12}$. They are said to be **pitch classes** because every note from $E$ to $D\mathrm{G}$ represents all multiples of those pitches.

As an example, take the $E$ major chord, $x=\{0,4,7\}$. If $E$ major is seen as a class of triads, it would be represented in the following ways:

$$E=[x]=\{[0],[4],[7]\}=\{\ldots,\{-12,-8,-5\},\{0,4,7\},\{12,16,19\},\ldots\}.$$

**Definition**

Let $x \in \mathrm{M}$, where $x=\{a,b,c\}$. A **transposition** is a function $T_n : \mathrm{M} \to \mathrm{M}$ given by

$$T_n(x)=T_n(\{a_1,a_2,a_3\})=x+n=\{a_1+n,a_2+n,a_3+n\},$$

where $n \in \mathrm{Z}$.

Although $T_n$ can only be applied to the 24 triads in $\mathrm{M}$, there is an infinite number of transpositions as $n \in \mathrm{Z}$. After having transposed a triad 12 times, the same sequence is obtained. For example,

$$T_0(E)=T_0(\{0,4,7\})=\{0,4,7\}$$
$$T_1(E)=T_1(\{0,4,7\})=\{1,5,8\}$$
$$\vdots$$
$$T_{12}(E)=T_{12}(\{0,4,7\})=\{0,4,7\}=T_0(E)$$
$$T_{13}(E)=T_{13}(\{0,4,7\})=\{1,5,8\}=T_1(E)$$
$$\vdots$$

**Example**

We will show that the operations on $T$ are well defined. That is, if $[x]$ is a triad of pitch classes in $\mathrm{M}$, for every $x_1,x_2 \in [x]$ we have $T_n(x_1) \equiv T_n(x_2)$.

Let $x_1,x_2 \in [x] \in \mathrm{M}$, where $x_1=\langle a_1,b_1,c_1\rangle$ and $x_2=\langle a_2,b_2,c_2\rangle$. Then $x_1$ and $x_2$ are elements in the class of triads $[x]=\langle [a],[b],[c]\rangle$. We see that $a_1,a_2 \in [a] \in \mathrm{Z}_{12}$, $b_1,b_2 \in [b] \in \mathrm{Z}_{12}$ and $c_1,c_2 \in [c] \in \mathrm{Z}_{12}$. Then

$$T_n(\langle a_1,b_1,c_1\rangle)=\langle a_1+n,b_1+n,c_1+n\rangle$$

and

$$T_n(\langle a_2,b_2,c_2\rangle)=\langle a_2+n,b_2+n,c_2+n\rangle.$$

Since $a_1 \in [a]$ and $a_2 \in [a]$,

$$(a_1 + n) \in [a+n] \text{ and } (a_2 + n) \in [a+n].$$

Similarly,

$$(b_1 + n), (b_2 + n) \in [b+n] \text{ and } (c_1 + n), (c_2 + n) \in [c+n].$$

Then

$$\begin{aligned} T_n(x_1) &= T_n(\langle a_1, b_1, c_1 \rangle) \\ &= \langle a_1 + n, b_1 + n, c_1 + n \rangle \\ &\equiv \langle a_2 + n, b_2 + n, c_2 + n \rangle = T_n(\langle a_2, b_2, c_2 \rangle) = T_n(x_2). \end{aligned}$$

**Definition**

Let $x \in \mathrm{M}$, where $x = \{X_1, X_2, X_3\}$. An **inversion** is a function $I_n : \mathrm{M} \to \mathrm{M}$ given by

$$I_n(x) = I_n(\{X_1, X_2, X_3\}) = -x + n = \{-X_1 + n, -X_2 + n, -X_3 + n\},$$

where $n \in \mathrm{Z}$.

As in the case of the transpositions, there are 24 triads to invert and an infinite number of inversions of each triad. However, when we invert a triad and then transpose it 12 times, the same sequence is obtained. For example,

$$\begin{gathered} I_0(E) = I_0(\{0,4,7\}) = \{0,8,5\} \\ I_1(E) = I_1(\{0,4,7\}) = \{1,9,6\} \\ \vdots \\ I_{12}(E) = I_{12}(\{0,4,7\}) = \{0,8,5\} = I_0(E) \\ I_{13}(E) = I_{13}(\{0,4,7\}) = \{1,9,6\} = I_1(E) \\ \vdots \end{gathered}$$

**Example**

We will show that the operations on $I$ are well defined. That is, if $[x]$ is a triad of pitch classes in $\mathrm{M}$, for every $x_1, x_2 \in [x]$ we have $I_n(x_1) \equiv I_n(x_2)$.

Let $x_1, x_2 \in [x] \in \mathrm{M}$, where $x_1 = \langle a_1, b_1, c_1 \rangle$ and $x_2 = \langle a_2, b_2, c_2 \rangle$. Then $x_1$ and $x_2$ are elements in the class of triads $[x] = \langle [a],[b],[c] \rangle$. We see that $a_1, a_2 \in [a] \in_{12}$, $b_1, b_2 \in [b] \in \mathrm{Z}_{12}$ and $c_1, c_2 \in [c] \in \mathrm{Z}_{12}$. Then

$$I_n(\langle a_1, b_1, c_1 \rangle) = \langle -a_1 + n, -b_1 + n, -c_1 + n \rangle$$

and

$$I_n(\langle a_2, b_2, c_2 \rangle) = \langle -a_2 + n, -b_2 + n, -c_2 + n \rangle.$$

Since $-a_1 \in [a]$ and $-a_2 \in [a]$,

$$(-a_1 + n) \in [a+n] \text{ and } (-a_2 + n) \in [a+n].$$

Similarly,

$$(-b_1 + n), (-b_2 + n) \in [b+n] \text{ and } (-c_1 + n), (-c_2 + n) \in [c+n].$$

Then

$$\begin{aligned} I_n(x_1) &= I_n(\langle a_1, b_1, c_1 \rangle) \\ &= \langle -a_1 + n, -b_1 + n, -c_1 + n \rangle \\ &\equiv \langle -a_2 + n, -b_2 + n, -c_2 + n \rangle \\ &= I_n(\langle a_2, b_2, c_2 \rangle) \\ &= I_n(x_2). \end{aligned}$$

**Definition**

The set of all the transposition and inversion functions is denoted as $TI$, and is defined as

$$TI = \{T_n, I_n \mid n = 0, \ldots, 11\}.$$

**Lemma**

In the $TI$ set, there exist the following relations:

(i) $T_m \circ T_n = T_{m+n \text{ mod}12}$,

(ii) $T_m \circ I_n = I_{m+n \text{ mod}12}$,

(iii) $I_m \circ T_n = T_{m-n \text{ mod}12}$,

(iv) $I_m \circ I_n = T_{m-n \text{ mod}12}$.

**Proof:**

(i)
$$\begin{aligned} T_m \circ T_n &= (T_m \circ T_n)(\{a,b,c\}) \\ &= T_m(T_n(\{a,b,c\})) \\ &= T_m(\{a+n, b+n, c+n\}) \\ &= \{a+n+m, b+n+m, c+n+m\} \\ &= \{a+(n+m), b+(n+m), c+(n+m)\} \\ &= T_{m+n \text{ mod}12}. \end{aligned}$$

(ii)
$$\begin{aligned} T_m \circ I_n &= (T_m \circ I_n)(\{a,b,c\}) \\ &= T_m(I_n(\{a,b,c\})) \\ &= T_m(\{-a+n, -b+n, -c+n\}) \\ &= \{-a+n+m, -b+n+m, -c+n+m\} \\ &= \{-a+(n+m), -b+(n+m), -c+(n+m)\} \\ &= I_{m+n \text{ mod}12}. \end{aligned}$$

(iii)
$$\begin{aligned} I_m \circ T_n &= (I_m \circ T_n)(\{a,b,c\}) \\ &= I_m(T_n(\{a,b,c\})) \\ &= I_m(\{a+n, b+n, c+n\}) \\ &= \{-(a+n)+m, -(b+n)+m, -(c+n)+m\} \\ &= \{-a+(m-n), -b+(m-n), -c+(m-n)\} \\ &= I_{m-n \text{ mod}12}. \end{aligned}$$

(iv)
$$\begin{aligned} I_m \circ I_n &= (I_m \circ I_n)(\{a,b,c\}) \\ &= I_m(I_n(\{a,b,c\})) \\ &= I_m(\{-a+n, -b+n, -c+n\}) \\ &= \{-(-a+n)+m, -(-b+n)+m, -(-c+n)+m\} \\ &= \{a+(m-n), b+(m-n), c+(m-n)\} \end{aligned}$$

$$= T_{m-n \bmod 12}.$$ □

**Proposition**

For every $n, k \in \mathbb{Z}$ such that $n \equiv k \bmod 12$, $T_n = T_k$ and $I_n = I_k$.

**Proof:**

As $n \equiv k \bmod 12$, then $n = 12q + k$ for some $q \in \mathbb{Z}$. Hence

$$T_n = T_{12q+k} = T_{12q} \circ T_k = (T_0)^q \circ T_k = (i)^q \circ T_k = T_k,$$

where $i$ is the identity transformation (or the translation by 0) and

$$I_n = I_{12q+k} = T_{12q} \circ I_k = (T_0)^q \circ I_k = (i)^q \circ I_k = I_k.$$ □

**Theorem**

The $TI$ set forms a group under composition.

**Proof:**

Clearly $TI$ is nonempty. For all $f, g \in TI$, by the above lemma,

$$f \circ g = h \in TI.$$

Therefore $TI$ is closed under composition.

By the properties of the composition of functions, the operation $\circ$ is associative. i.e., for each $f, g, h \in TI$,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Next, the following is satisfied:

$$T_0 \circ T_n = T_{0+n} = T_n,$$
$$T_n \circ T_0 = T_{n+0} = T_n,$$
$$T_0 \circ I_n = I_{0+n} = I_n,$$
$$I_n \circ T_0 = I_{n-0} = I_n.$$

Hence $T_0 \in TI$ is the identity element.

On the one hand the relationships:

$$T_n \circ T_{12-n} = T_{n+12-n} = T_{12} = T_0$$

and

$$T_{12-n} \circ T_n = T_{12-n+n} = T_{12} = T_0$$

implies

$$T_n^{-1} = T_{12-n}.$$

On the other we have

$$I_n \circ I_n = T_{n-n} = T_0$$

and this shows that $I_n^{-1} = I_n$. Hence $TI$ is a group under composition. □

## Conclusion

Musical and mathematical notions are brought together, such as scales and modular arithmetic, octave identification and equivalence relation, intervals and logarithms, tone and trigonometry, equal temperament and exponents, overtones and integers, timbre and harmonic analysis, tuning and rationality etc. Each of these notions enters the scene because it is involved in one way or another with a point where mathematics and music converge. By examining similarities between the two subjects on many different levels, from infant development, to how the brain works with pattern, to the level of abstraction, to creativity and beauty, we will arrive at the ultimate connection between the subjects; that similar patterns of thought underlies both mathematics and music.

### References

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